Chapter 6

Conservation in Quantum Mechanics, and Beyond Hodge's Theorem

6.1 Introduction

In this chapter, we deal with results in physics and mathematics that arise from Kähler's first Green's identity for differential forms. On the left hand side of this identity, there is a exterior differentiation. If the right hand side is zero, so is the left hand side and, by Stoke's theorem, the conservation law of the exterior calculus follows. We shall see that, in the KC as in Dirac's theory, the right hand side of the Green identity that gives rise to the conservation law is quadratic in the wave function or, more precisely, linear in it and its conjugate. Of great interest is the specialization of this right hand side when we write the wave function as a sum of two parts, respectively associated with the two ideals in the algebra that are defined by time translation symmetry.

When the coupling is electromagnetic, what is conserved – aside from energy – is some magnitude which comes with both signs. The magnitude is conserved, not what each of the parts refers to. The obvious interpretation or the conserved magnitude is charge, not probability, which does not come with both signs and can thus only be a derived concept. This implies a radically new vision of quantum physics.

Another important application of Green's first identity is a uniqueness theorem for a differential form on a manifold (or region thereof) when both its exterior and interior derivatives are given, as well as the specification of the differential form at the boundary. This theorem is instrumental in obtaining a Helmholtz theorem for differential $r-$forms in Euclidean spaces E_n and in regions or r-surfaces thereof. This results immediately generalizes by embedding to any differentiable manifold, Riemannian ones in particular, that can be embedded in Euclidean spaces.

The generalization of Helmholtz theorem is at the same time a generalization of Hodge's decomposition theorem, since one not only derives the latter theorem, but one actually obtains their forms as integrals, in the guise of Helmholtz theorem. In the Helmholtz theorem of the vector calculus, as in its translation to differential forms and generalization to any grade of the differential form and any dimension of the Euclidean space, one of the two terms is closed and the other one is co-closed. The appearance in Hodge's theorem of a term – the harmonic one – is due to the fact that certain terms that cancel at the boundary under the more restricted conditions of applicability of Helmholtz theorem no longer do so.

In conclusion, high power results in both physics and mathematics arise from Green's first identity of the Kähler calculus.

6.2 Green's identities

Kähler defines scalar products of different grades for arbitrary differential forms. For arbitrary differential forms, they are written as $(u, v)_r$, where r is the complement to n of the grade of the product. The notations $(u, v)_0$ and $(u, v)_1$ will be used to refer to scalar products of grades n and $n-1$ respectively, which we are about to define. We shall confine ourselves to defining scalar products of those grades, which are the only ones that enter Kähler's Green identity. Due to the fact that there is no possibility for confusion, Kähler uses the symbol (u, v) for what we have momentarily called $(u, v)_0$, and reserves the subscript zero to refer to the 0-grade part of differential forms of, in general, "inhomogeneous grade".

Following Kähler, we shall use the symbol ζ for the operator that reverses the order of vectors in all products. The scalar product of grade n, or simply scalar product, is defined as

$$(u, v) = (\zeta u \vee v) \wedge z, \tag{6.1}$$

although, as Kähler himself points out, it is better to use the term scalar product to refer to the actual evaluation, i.e. integration, of the differential form (u, v). Since the right hand side of (6.1) obviously vanishes unless $\zeta u \vee v$ is a 0-form, it can be rewritten as

$$(u, v) = (\zeta u \vee v)_0 z = (\zeta u \vee v) \wedge z. \tag{6.2}$$

One similarly defines:

$$(u, v)_1 = e_i(dx^i \vee u, v) = e_i[(\zeta u \vee dx^i \vee v) \wedge z]. \tag{6.3}$$

We shall later use that

$$(u, v) = (v, u) \tag{6.4}$$

i.e. that $(\zeta u \vee v)_0$ equals $(v \vee \zeta u)_0$, which is trivial.

We shall also use that

$$de_i = d_i - e_i d. \tag{6.5}$$

Indeed

$$de_i u = de_i(dx^i \wedge u' + u'') = du' \tag{6.6}$$

$$d_i u = d_i(dx^i \wedge u' + u'') = dx^i \wedge d_i u' + d_i u'', \tag{6.7}$$

and

$$-e_i du = -e_i(-dx^i \wedge du' + du'') = du' - dx^i \wedge e_i du' - e_i du''. \tag{6.8}$$

But

$$e_i du' = d_i u', \qquad e_i du'' = d_i u''. \tag{6.9}$$

Using (6.9) in (6.8) and bringing (6.6), (6.7) and (6.8) together, we get (6.5).

We now prove Green's first identity, which reads

$$d(u, v)_1 = (\partial u, v) + (u, \partial v). \tag{6.10}$$

We differentiate (6.3), use (6.5) and the fact that the square bracket on which the operator d is acting is a differential n-form and that, therefore, its exterior derivative is zero. We thus have:

$$\begin{aligned} d(u, v)_1 &= de_i[(\zeta u \vee dx^i \vee v) \wedge z] = d_i[(\zeta u \vee dx^i \vee v) \wedge z] - \\ -e_i d[(\zeta u \vee dx^i \vee v) \wedge z] &= d_i[(\zeta u \vee dx^i \vee v) \wedge z]. \end{aligned} \tag{6.11}$$

But $d_i z = 0$. Hence

$$d(u, v)_1 = d_i(...) \wedge z = (d_i \zeta u) \vee dx^i \vee v \wedge z + (\zeta u \vee dx^i \vee d_i v) \wedge z, \tag{6.12}$$

where we have used the distributive property of the d_i operator and that d_i of dx_i is zero, where "..." stands for $\zeta u \vee dx^i \vee v$.

We now use that

$$(d_i \zeta u) \vee dx^i = (\zeta d_u u) \vee dx^i = \zeta(dx^i \vee d_i u) = \zeta \partial u. \tag{6.13}$$

The first term on the right hand side of (6.10) follows from the first term on the right hand side of (6.12). For the second term, just notice that $dx^i \wedge d_i v$ equals ∂v. End of proof.

A second Green identity results by first replacing first v and later u with ∂v and ∂u respectively. Thus

$$d(u, \partial v)_1 = (\partial u, \partial v) + (u, \Delta v), \tag{6.14}$$
$$d(v, \partial u)_1 = (\partial v, \partial u) + (v, \Delta u). \tag{6.15}$$

where Δ stands for $\partial\partial$. Subtracting (6.15) from (6.14), we get

$$(u, \Delta u) - (v, \Delta u) = d[(u, \partial v)_1 - (v, \partial u)_1]. \tag{6.16}$$

Needless to say that there are other Green identities, like, for instance, if we replace both u and v with ∂u and ∂v.

6.3 The two signs of charge

6.3.1 The conjugate Kähler equation

The first Green identity prompts us to find a conjugate Kähler equation such that its solutions v will give rise to a conservation law through scalar multiplication with the solutions u of the "direct Kähler equation". We seek it in the form $\partial u = bu$, and try to find b as a function of a. We do not yet assume electromagnetic coupling. We shall solve the equation

$$(u, \partial v) = -(v, \partial u). \tag{6.17}$$

We have

$$(u, \partial v) = (\partial v, u) = (bv, u) = [\zeta v \vee \zeta b \vee u] \wedge z = (v, \zeta b \vee u). \tag{6.18}$$

For (6.18) to become (6.17), we want

$$(\zeta b)u = -\partial u = -au. \tag{6.19}$$

Hence

$$b = -\zeta a, \tag{6.20}$$

and the conjugate Kähler equation therefore is

$$\partial u = -(\zeta a)u. \tag{6.21}$$

We have thus shown that, if u and v are respective solutions of a direct and its conjugate Kähler equation, then $d(u,v)_1 = 0$ because $(\partial u, v)+(u,\partial v) = 0$.

6.3.2 The electromagnetic conservation law

We now show that if u is a solution of the electromagnetic Kähler equation, $\eta\overline{u}$ is a solution of its conjugate equation. For electromagnetic coupling,

$$a = iE_0 + e\phi, \tag{6.22}$$

with $e = \mp|e|$ and with ϕ as the electromagnetic 1-form. Let overbar denote complex conjugation. Since $\partial\eta = -\eta\partial$ and $\eta\bar{a} = a = \zeta a$, we have

$$\partial(\eta\overline{u}) = -\eta\partial\overline{u} = -\eta\overline{\partial u} = -\eta(\overline{a}\vee\overline{u}) = -a\vee\eta u = -(\zeta a)\vee\eta u. \tag{6.23}$$

The conservation law then takes the form

$$d(u,\eta\overline{v})_1 = 0, \tag{6.24}$$

and, in particular,

$$d(u,\eta\overline{u})_1 = 0. \tag{6.25}$$

6.3.3 Computations with scalar products

We produce some formulas needed for the computation of $(u,\eta\overline{u})_1$ when we split u into members of complementary ideals associated with time translation symmetry.

Taking into account (6.4), the equation

$$(\zeta u,\zeta v) = (u,v) \tag{6.26}$$

readily follows since

$$(\zeta u,\zeta v) = (u\vee\zeta v)_0 z = (u\vee v)_0 z = (u,v). \tag{6.27}$$

(6.26) is used in

$$
\begin{aligned}
(u \vee w, v) = (w, \zeta u \vee v) = (\zeta u \vee v, w) = (\zeta(\zeta u \vee v), \zeta w) = \\
= (\zeta u \vee v \vee \zeta w)_0 z = (u, v \vee \zeta w),
\end{aligned}
\tag{6.28}
$$

where the first step follows from the definition of scalar product of grade n; we have then used (6.4) and (6.26).

Since $\zeta(w, u) \vee v = \zeta u \vee (\zeta u \vee v)$, it readily follows that

$$(w \vee u, v) = (u, \zeta w \vee v), \tag{6.29}$$

which is in turn used to obtain

$$(dx^\mu \vee u, v) = (u, \zeta dx^\mu \vee v) = (u, dx^\mu \vee v) = (dx^\mu \vee v, u) \tag{6.30}$$

and, therefore,

$$(v, u)_1 = (u, v)_1. \tag{6.31}$$

We shall later need

$$(u \vee w, v)_1 = (u, v \vee \zeta w)_1, \tag{6.32}$$

which we prove as follows

$$
\begin{aligned}
(u \vee w, v)_1 &= e_\mu\{[\zeta(u \vee w) \vee dx^\mu \vee v] \wedge z\} = e_\mu(u \vee w, dx^\mu \vee v) \\
&= e_\mu(u, dx^\mu \vee v \vee \zeta w) = (u, v \vee \zeta w)_1,
\end{aligned}
\tag{6.33}
$$

where we have used (6.28).

We are using Greek symbols to emphasize that we are not restricting ourselves to 3-space. We shall later use spacetime indices (Greek) and 3-space indices (Latin) in the same argument.

6.3.4 The current $(u, \overline{\eta v})_1$ in terms of elements of the ideals generated by $\varepsilon^\pm$

Recall

$$u = {}^+u \vee \epsilon^+ + {}^-u \vee \epsilon^-, \quad v = {}^+v \vee \epsilon^+ + {}^-v \vee \epsilon^-. \tag{6.34}$$

In the next few lines, let ϵ be ϵ^+ or ϵ^- but not both at the same time. Since ϵ is an idempotent, $\epsilon \vee \epsilon = \epsilon$. Also, $\zeta\epsilon = \epsilon$. Hence

$$(u \vee \epsilon, \eta\overline{v} \vee \overline{\epsilon})_1 = (u \vee \epsilon \vee \epsilon, \eta\overline{v} \vee \overline{\epsilon})_1 = (u \vee \epsilon, \eta\overline{v} \vee \overline{\epsilon} \vee \zeta\epsilon)_1 = 0 \tag{6.35}$$

where we have used (6.32) and that

$$\bar{\epsilon} \vee \zeta\epsilon = \bar{\epsilon} \vee \epsilon = 0 \tag{6.36}$$

since $\bar{\epsilon}^{\pm} = \epsilon^{\mp}$ and $\epsilon^+\epsilon^- = \epsilon^-\epsilon^+ = 0$.

In order to simplify notation, let us define

$$[u, v] = (u, \eta\bar{v})_1. \tag{6.37}$$

We shall now use (6.34), (6.35) and (.6.37) to obtain

$$[u, v] = [{}^+u \vee \epsilon^+, {}^+v \vee \epsilon^+] + [{}^-u \vee \epsilon^-, {}^-v \vee \epsilon^-]. \tag{6.38}$$

The ${}^+u$, ${}^-u$, ${}^+v$, ${}^-v$ are spatial differentials since the dt dependence of u and v has been replaced through $dt = (1/i)(\epsilon^+ - \epsilon^-)$. They are not "strict" since, in general, they will depend on t. Let such differentials be represented as p and q. The following then applies to both terms in (6.38)

$$4[p \vee \epsilon^{\pm}, q \vee \epsilon^{\epsilon}] = [p, q] \mp [p, q \vee idt] \mp [p \vee idt, q] + [p \vee idt, q \vee idt]. \tag{6.39}$$

Using (6.31), we readily prove that the first and second terms are respectively equal to the fourth and third terms. Hence

$$[p \vee \epsilon^{\pm}, q \vee \epsilon^{\pm}] = \frac{1}{2}[p, q] \ \mp p, q \vee idt]. \tag{6.40}$$

6.3.5 The emergence of the terms in the continuity equation

We proceed to develop $[p, q]$. For this purpose, we notice that $(\zeta p \vee dt \vee \eta\bar{q})_0 = 0$, since there is no dt factor in ζp and $\eta\bar{q}$. We also notice an alternative way of writing $(u, v)_1$

$$(u, v)_1 = e_\mu(dx^\mu \vee u, v) = e_\mu[(\zeta u \vee dx^\mu \vee v)_0 z] = (\zeta u \vee dx^\mu \vee v)_0 e_\mu z. \tag{6.41}$$

Let z and w represent the unit differential 4-form and 3-form respectively. Then

$$e_k z = e_k(w \wedge idt) = e_k w \wedge idt. \tag{6.42}$$

Hence

$$[p, q] = [(\zeta p \vee dx^k \vee \eta\bar{q})_0 e_k w] \wedge idt = \{p, \eta\bar{q}\}_1 \wedge idt, \tag{6.43}$$

where $\{p, \eta\overline{q}\}_1$ is the symbol used to represent the scalar product of grade $n-1$ in the Kähler algebra for $n = 3$.

We similarly have

$$[p, q \vee idt] = (\zeta p \vee dt \vee \eta\overline{q} \vee idt)_0 e_t z = -i(\zeta p \vee dt \vee dt \vee \overline{q})_0 wi = \\ = -(\zeta p \vee \overline{q})_0 w = -\{p, \overline{q}\}. \tag{6.44}$$

Notice the presence of the first dt, instead of dx^i, inside the parenthesis, based on the same type of argument as before.

In view of (6.43) and (6.44), Eq. (6.39) now reads

$$2[p \vee \varepsilon^{\pm}, q \vee \varepsilon^{\pm}] = \{p, \eta\overline{q}\}_1 \wedge idt + pm\{p, \overline{q}\}. \tag{6.45}$$

We next make $q = p = {}^{+}u$ in (6.46) and obtain

$$[{}^{+}u \vee \varepsilon^{+}, {}^{+}u \vee \varepsilon^{+}] = \frac{1}{2}\{{}^{+}u, {}^{+}\overline{u}\} + \frac{1}{2}\{{}^{+}u, \eta^{+}u\}_1 \wedge idt. \tag{6.46}$$

Next we make $q = p = {}^{-}u$ and obtain an equation almost equal in form to (6.46), except that, in addition to the replacement ${}^{+} \rightarrow {}^{-}u$, there will be a change in sign in one of the terms. Thus

$$\begin{aligned}[u, u] &= [{}^{+}u \vee \varepsilon^{+}, {}^{+}u \vee \varepsilon^{+}] + [{}^{-}u \vee \varepsilon^{-}, {}^{-}u \vee \varepsilon^{-}] = \\ &= \frac{1}{2}\{{}^{+}u, {}^{+}\overline{u}\} + \frac{1}{2}\{{}^{+}u,\ \eta^{+}\overline{u}\}_1 \wedge idt \\ &- \frac{1}{2}\{{}^{-}u, {}^{-}\overline{u}\} + \frac{1}{2}\{{}^{-}u,\ \eta^{-}\overline{u}\}_1 \wedge idt.\end{aligned} \tag{6.47}$$

Each of the two lines has the form of a scalar-valued space time current where the spatial 3-forms are volume densities, ρw, and where the spacetime 3-forms are the currents in the sense similar to the "vector current".

In section (§15) of his 1962 paper, Kähler had already made the remark that, when the metric is positive definite, the product (u, u) for arbitrary u is a number that is everywhere ≥ 0 times the volume differential, and it is positive definite at every point P where $u(P) \neq 0$. In (6.47), the metric at work is the Euclidean metric and, therefore, both $\{{}^{+}u, {}^{+}\overline{u}\}$ and $\{{}^{-}u, {}^{-}\overline{u}\}$ are nowhere negative.

In view of the foregoing considerations and of Eq. (6.47), Kähler concludes that the characterization of the negative electrons state differential form u by $u \vee \varepsilon^{-} = u$, $u \vee \varepsilon^{+} = 0$ brings about a density

$$\rho w = -\frac{|e|}{2}\{{}^{-}u, {}^{-}u\}$$

with $\rho \leq 0$ everywhere.

This is a tremendously important result for the foundations of quantum mechanics. It shows that, in Kähler's theory, the wave "function" is not a probability amplitude but, so to speak, a "charge amplitude".

6.4 A uniqueness for differential $k-$forms of definite grade under Helmholtz type conditions

Let R be a differentiable manifold and let ∂R be its boundary. Let (u_1, u_2) be differential $k-$forms in R such that $du_1 = du_2$, $\delta u_1 = \delta u_2$ on R, and that u_1 equals u_2 on ∂R. The uniqueness theorem states that the differential form is uniquely defined.

β defined as $u_1 - u_2$ satisfies

$$d\beta = 0 = \delta\beta \text{ on } R, \qquad\qquad \beta = 0 \text{ on } \partial R, \tag{6.48}$$

and, locally,

$$(\beta = d\alpha,\ \delta d\alpha = 0) \quad \text{on } R, \qquad\qquad d\alpha = 0 \quad \text{on } \partial R. \tag{6.49}$$

Equation (6.10) with $u = \alpha$ and $v = d\alpha$ reads

$$d(\alpha, d\alpha)_1 = (\alpha, \partial d\alpha) + (d\alpha, \partial\alpha). \tag{6.50}$$

We use (6.49) to obtain

$$(\alpha, \partial d\alpha) = (\alpha, dd\alpha) + (\alpha, \delta d\alpha) = 0 + 0. \tag{6.51}$$

Consider next $(d\alpha, \partial\alpha)$. If α is of definite grade, so are $d\alpha$ and $\delta\alpha$, but their grades differ by two units. Their scalar product is, therefore, zero. On the other hand, we have, with a_A defined by $d\alpha = a_A dx^A$ (with summation over the algebra as a module),

$$(d\alpha, \partial\alpha) = (d\alpha, \delta\alpha) + (d\alpha, d\alpha) = 0 + \sum |a_A|^2. \tag{6.52}$$

Substituting (6.51) and (6.52) in (6.50), applying Stokes theorem and using $d\alpha = 0$ on R, we get

$$\int_R \sum |a_A|^2 = \int_R d(\alpha, d\alpha)_1 = \int_{\partial R} (\alpha, d\alpha)_1 = 0. \tag{6.53}$$

Hence all the a_R's are zero in R itself and so is, therefore, α and β $(=d\alpha)$. It follows from the defintion of β as $u_1 - u_2$ that $u_1 = u_2$. The theorem has thus been prooved.

6.5 Helmholtz Theorems for $k-$forms

6.5.1 Helmholtz Theorem for $k-$forms in E_3

In this section, we shall try to avoid potential confusion by replacing the symbol z with the symbol w for the unit differential 3−form.

With $r_{12} \equiv [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}$, the standard Helmholtz theorem of the vector calculus states

$$\mathbf{v} = -\frac{1}{4\pi}\nabla \int_{E'_3} \frac{\nabla' \cdot \mathbf{v}(\mathbf{r}')}{r_{12}} dV' + \frac{1}{4\pi}\nabla \times \int_{E'_3} \frac{\nabla' \times \mathbf{v}(\mathbf{r}')}{r_{12}} dV'. \tag{6.54}$$

has an immediate translation to the Helmholtz theorem for differential 1−forms in E_3. It reads

$$\alpha = -\frac{1}{4\pi} d \int_{E'_3} \frac{(\delta'\alpha')w'}{r_{12}} - \frac{1}{4\pi}\delta \left(dx^j dx^k \int_{E'_3} \frac{d'\alpha' \wedge dx'^i}{r_{12}} \right), \tag{6.55}$$

This theorem is a particular case of the theorem proved in the next subsection. It implies a similar theorem for differential 2−forms, as follows from using that the equation $\alpha = w\beta$, uniquely defines β. We substitute it in (6.55) and solve for β:

$$\beta = \frac{1}{4\pi} wd \left(\int_{E'_3} \frac{\delta'(w'\beta')}{r_{12}} w' \right) + \frac{1}{4\pi} w\delta \left(dx^{jk} \int_{E'_3} \frac{d'(w'\beta') \wedge dx'^i}{r_{12}} \right). \tag{6.56}$$

Denote the first integral in (6.56) as I and the second one as I^i. We have $wdI = \delta(wI)$ and $w'\delta'(w'\beta') = w'(w'd\beta') = -d\beta' = -d\beta' \wedge 1$. The exterior product by 1 is superfluous, except for the purpose for making later Eq. (6.60) clear. Similarly, $w\delta(dx^{jk}I^i) = d(wdx^{jk}I^i) = -d(dx^i I^i)$ and

$$d'(w'\beta') \wedge dx'^i = dx'^i \wedge d'(w'\beta') = \frac{1}{2}\left[dx'^i w'\delta'\beta' + w'\delta'\beta' dx'^i\right] =$$

$$= \frac{1}{2}(dx'^{jk}\delta'\beta' + \delta'\beta' dx'^{jk}) = dx'^{jk} \wedge \delta'\beta' = \delta'\beta' \wedge dx'^{jk}. \tag{6.57}$$

We use these results in (6.56), change the order of the terms and get

$$\beta = -\frac{1}{4\pi} d \left(dx^i \int_{E_3'} \frac{\delta' \beta' \wedge dx'^{jk}}{r_{12}} \right) - \frac{1}{4\pi} \delta \left(w \int_{E_3'} \frac{\delta' \beta' \wedge 1}{r_{12}} \right), \tag{6.58}$$

Write the first term in (6.55) as

$$-\frac{1}{4\pi} d \left[1 \wedge \int_{E_3'} \frac{(\delta' \alpha') \wedge w'}{r_{12}} \right]. \tag{6.59}$$

Let the index A label a Cartesian basis of the algebra as module. Let $dx^{\bar{A}}$ be the unique element in the basis such that $dx^A \wedge dx^{\bar{A}} = w$. Define $\int_{E_3} \gamma_r$ if the grade r of γ is different from 3. All four terms on the right of (6.55) and (6.58) are thus of the form

$$-\frac{1}{4\pi} d \left[dx^A \int_{E_3'} \frac{(\delta' \_\_) \wedge dx'^{\bar{A}}}{r_{12}} \right] \quad \text{or} \quad -\frac{1}{4\pi} \delta \left[dx^A \int_{E_3'} \frac{(d' \_\_) \wedge dx'^{\bar{A}}}{r_{12}} \right]. \tag{6.60}$$

Take, for instance, the first of the two expressions in (6.60). We sum over all A, equivalently, over all $\bar{A}'$. The grade of $(\delta' \_\_)$ determines the grade of the only $dx'^{\bar{A}}$ that may yield not zero integral since the sum of the respective grades must be 3. For each surviving value of the index $\bar{A}$, the value of the index A —thus the specific dx^A at the front of the integral— is determined. We shall later show for ulterior generalization that we may replace the Cartesian basis with any other basis, which we shall choose to be orthonormal since they are the "canonical ones" of Riemannian spaces.

6.5.2 Helmholtz Theorem for Differential k-forms in E_n

Let ω^A $(\equiv \omega^{i_1}\omega^{i_2}...\omega^{i_r})$ denote elements of a basis in the Kähler algebra of differential forms such that the ω^μ are orthonormal. The purpose of using an orthonormal basis is that exterior products can be replaced with Clifford products. Let $\omega^{\bar{A}}$ be the monomial (uniquely) defined by $\omega^A \omega^{\bar{A}} = z$, with no sum over repeated indices.

The generalized Helmholtz theorem in E_n reads as follows

$$\alpha = -\frac{1}{(n-2)S_{n-1}} [d(\omega^A I_A^\delta) + \delta(\omega^A I_A^d)], \tag{6.61}$$

with summation over a basis in the algebra and where

$$I_A^\delta \equiv \int_{E'_n} \frac{(\delta'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{n-2}}, \qquad I_A^d \equiv \int_{E'_n} \frac{(d'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{n-2}}. \tag{6.62}$$

r_{12} is defined by $r_{12}^2 = (x_1 - x'_1)^2 + ... + (x_n - x'_n)^2$ in terms of Cartesian coordinates.

It proves convenient for performing differentiations to replace ω^i, ω^A and $\omega^{\bar{A}}$ with dx^i, dx^A and $dx^{\bar{A}}$. If the results obtained are invariants, one can re-express the results in terms of arbitrary bases.

We proceed again via the uniqueness theorem, as in the vector calculus, with specification now of $d\alpha$, $\delta\alpha$ and that α goes sufficiently fast at ∞. vanishing of α at infinity. Because of the annulment of dd and $\delta\delta$, the proof reduces to showing that $\delta d(dx^A I_A^\delta)$ and $d\delta(dx^A I_A^d)$ respectively yield $\delta\alpha$ and $d\alpha$, up to the factor at the front in (6.61). Since the treatment of both terms is the same, we shall carry them in parallel, as in

$$\begin{pmatrix} \delta \\ d \end{pmatrix} \alpha \rightarrow \begin{pmatrix} \delta d \\ d\delta \end{pmatrix} dx^A I_A^{\binom{\delta}{d}} = \partial\partial dx^A I_A^{\binom{\delta}{d}} - \begin{pmatrix} d\delta \\ \delta d \end{pmatrix} dx^A I_A^{\binom{\delta}{d}}. \tag{6.63}$$

In the first term on the right hand side of (6.63), we move $\partial\partial$ to the right of dx^A, insert it inside the integral with primed variables, multiply by $-\frac{1}{(n-2)S_{n-1}}$ and treat the integrand as a distribution. We easily obtain that the first term yields $\binom{\delta\alpha}{d\alpha}$.

For the last term in (6.63), we have

$$\begin{pmatrix} d\delta \\ \delta d \end{pmatrix} dx^A I_A^{\binom{\delta}{d}} = \begin{pmatrix} d\left[dx^i \cdot dx^A \frac{\partial I_A^\delta}{\partial x^i}\right] \\ \delta\left[(\eta dx^A) \wedge dx^i \frac{\partial I_A^d}{\partial x^i}\right] \end{pmatrix}. \tag{6.64}$$

For the first line in (6.64), we have used that $d_h u = \frac{\partial}{\partial x^h}$ in Cartesian coordinates, and that $\delta u = dx^h \cdot d_h u$. For the development of the second line, we have used the Leibniz rule.

We use the same rule to also transform the first line in (6.64),

$$d\left(dx^i \cdot dx^A \frac{\partial I_A^\delta}{\partial x^i}\right) = [\eta(dx^i \cdot dx^A)] \wedge dx^l \frac{\partial^2 I_A^\delta}{\partial x^l \partial x^i} = (dx^A \cdot dx^i) \wedge dx^l \frac{\partial^2 I_A^\delta}{\partial x^l \partial x^i}. \tag{6.65}$$

For the second line, we get

$$\delta\left[(\eta dx^A)\wedge dx^i\frac{\partial I_A^d}{\partial x^i}\right]=dx^l\cdot\left[\frac{\partial^2 I_A^d}{\partial x^l\partial x^i}(\eta dx^A)\wedge dx^i\right]. \tag{6.66}$$

We shall use here that

$$dx^l[(\eta dx^A)\wedge dx^i]=-\eta[\eta(dx^A\wedge dx^i)]\cdot dx^l=(dx^A\wedge dx^i)\cdot dx^l, \tag{6.67}$$

thus obtaining

$$\delta\left[(\eta dx^A)\wedge dx^i\frac{\partial I_A^d}{\partial x^i}\right]=(dx^A\wedge dx^i)\cdot dx^l\frac{\partial^2 I_A^d}{\partial x^l\partial x^i}. \tag{6.68}$$

Getting (6.65) and (6.66) into (6.64), we obtain

$$\begin{pmatrix} d\delta \\ \delta d\end{pmatrix} dx^A I_A^{\binom{\delta}{d}}=\left[dx^A(\,{}^{\cdot}_{\wedge})dx^i\right](\,{}^{\wedge}_{\cdot})dx^l\int_{E_n'}\frac{\partial^2}{\partial x'^i\partial x'^l}\frac{1}{r_{12}^{n-2}}\begin{pmatrix}\delta'\alpha'\\ d'\alpha'\end{pmatrix}\wedge dx'^{\bar{A}}. \tag{6.69}$$

Integration by parts with respect to x'^i yields two terms. The total differential term is

$$\left[dx^A(\,{}^{\cdot}_{\wedge})dx^i\right](\,{}^{\wedge}_{\cdot})dx^l\int_{E_n'}\frac{\partial}{\partial x'^i}\left[\left(\frac{\partial}{\partial x'^l}\frac{1}{r_{12}^{n-2}}\right)\begin{pmatrix}\delta'\alpha'\\ d'\alpha'\end{pmatrix}\wedge dx'^{\bar{A}}\right]. \tag{6.70}$$

Application of Stokes theorem yields

$$\left[dx^A(\,{}^{\cdot}_{\wedge})dx^i\right](\,{}^{\wedge}_{\cdot})dx^l\int_{\partial E_n'}\left(\frac{\partial}{\partial x'^l}\frac{1}{r_{12}^{n-2}}\right)\left\{dx'^i\cdot\left[\begin{pmatrix}\delta'\alpha'\\ d'\alpha'\end{pmatrix}\wedge dx'^{\bar{A}}\right]\right\}, \tag{6.71}$$

where we have indulged in the use of parentheses for greater clarity. This term is null if the differentiations of α go sufficiently fast to zero at infinity.

The other term resulting from the integration by parts is

$$-\left[dx^A(\,{}^{\cdot}_{\wedge})dx^i\right](\,{}^{\wedge}_{\cdot})dx^l\int_{E_n'}\left(\frac{\partial}{\partial x'^l}\frac{1}{r_{12}^{n-2}}\right)\frac{\partial}{\partial x'^i}\left(\begin{pmatrix}\delta'\alpha'\\ d'\alpha'\end{pmatrix}\wedge dx'^{\bar{A}}\right). \tag{6.72}$$

This is zero because of cancellations that take place in groups of three different indices, as shown in the next subsection.

In terms of Cartesian bases, we have, on the top line of the left hand side of (6.63)

$$dx^A\int_{E_n'}\frac{(\delta'\alpha')\wedge dx'^{\bar{A}}}{r_{12}^{n-2}}. \tag{6.73}$$

It is preceded by invariant operators, which we may ignore for present purposes. We move dx^A inside the integral, where we let $(\delta'\alpha')_A$ be the notation for the coefficients of $\delta'\alpha'$. We thus have, for that first term,

$$\int_{E'_n} \frac{dx^A \wedge [(\delta'\alpha')_A dx'^A] \wedge dx'^{\bar{A}}}{r_{12}^{n-2}}. \tag{6.74}$$

The numerator can be further written as $(\delta'\alpha')_A dx^A z'$. It is clear that z and $(\delta'\alpha')_A dx'^A$ are invariants, but not immediately clear that $(\delta'\alpha')_A dx^A$ also is so. Whether we have the basis dx^A or dx'^A as a factor is immaterial. since the invariance of $(\delta'\alpha')_A dx'^A$ can be seen as following from the matching of the transformations of $(\delta'\alpha')_A$ and dx'^A each in accordance with its type of covariance. The same matching applies if we replace ω'^A with dx^A, since dx'^A and dx^A transform in unison.

We have shown that (6.61)-(6.62) constitutes the decomposition of α into closed and co-closed differential forms. It solves the problem of integrating the system $d\alpha = \mu$, $\delta\alpha = \nu$, for given μ and ν, and with the stated boundary condition

6.5.3 Identical vanishing of some integrals

As we are about to show, expressions (6.72) cancel identically (Notice that (6.71) cancels at infinity for fast vanishing; identical vanishing is not needed).

Consider the first line in (6.72). Let α be of grade $h \geq 2$ (If h were one, the dot product of dx^A with dx^i would be zero). Let p and q be a specific pair of indices in a given term in α, i.e. in its projection $a'_{pqC},_{pq}\, dx^A$ upon some specific basis element dx^A. Such a projection can be written as

$$(a'_{pqC} dx'^p \wedge dx'^q \wedge dx'^C,$$

where dx'^A is a unit monomial differential $1-$form (there is no sum over repeated indices. We could also have chosen to write the same term as

$$(a'_{qpC} dx'^q \wedge dx'^p \wedge dx'^C,$$

with $a'_{qpC} = -a'_{pqC}$. Clearly, dx'^C is uniquely determined if it is not to contain dx^p and dx^q. We then have

$$\delta'(a_{'pqC} dx'^p \wedge dx'^q \wedge dx'^C) = a'_{pqC},_p\, dx'^q \wedge dx'^C - a_{pqC},_q\, dx'^p \wedge dx'^C. \tag{6.75}$$

The two terms on the right are two different differential 2−forms. They enter two different integrals, corresponding to $dx'^q \wedge dx'^C$ and $dx'^p \wedge dx'^C$ components of $\delta'\alpha'$. To avoid confusion, we shall refer to the basis elements in the integrals as dx'^B since they are $(h-1)-$forms, unlike the dx'^A of (6.75), which are differential $h-$forms

When taking the first term of (6.75) with $i = p$ into the top line of (6.72), the factor at the front of the integral is

$$-(dx^B \cdot dx^p) \wedge dx^l.$$

But this factor is zero since dx^B is $dx^q \wedge dx^C$, which does not contain dx^P as a factor. Hence, for the first term, we need only consider $i = q$. By the same argument, we need only consider $i = q$ for the second term in (6.75). Upon multiplying the dx'^B's by pertinent dx'^B's, we shall obtain the combination

$$(a'_{pqC},_{pq} - a'_{pqC},_{qp})z'$$

as a factor inside the integral for the first line of (6.72). We could make this statement because the factor outside also is the same one for both terms: $(dx^q \wedge dx^C) \cdot dx^q$ and $(dx^p \wedge dx^C) \cdot dx^p$ are equal. The contributions arising from the two terms on the right hand side of (6.75) thus cancel each other out. We would proceed similarly with any other pair of indices, among them those containing either p or q. The annulment of the top line of (6.72) has been proved.

In order to prove the cancellation of the second line in (6.72), the following considerations will be needed. A given dx^A determines its corresponding $dx'^{\bar{A}}$, and vice versa. It follows then that only the term proportional to dx'^A in $d'\alpha'$ exterior multiplies $dx'^{\bar{A}}$, which is of the same grade as $d'\alpha'$, i.e. $h + 1$. Hence $dx^A \wedge dx^i$ is of grade 3 or greater for $h > 0$. If $dx^A \wedge dx^i$ is not to be null, dx^i cannot be in dx^A. Hence, $dx'^{\bar{A}}$ contains dx^i as a factor.

Let (p, q, r) be a triple of three different indices in $dx^A \wedge dx^i$. When i is p or q or r, the respective pairs (q, r), (r, p) and (p, q) are in dx^A. We may thus write

$$dx'^A = dx'^C \wedge dx'^q \wedge dx'^r, \qquad dx'^{\bar{A}} = dx'^p \wedge dx'^B. \tag{6.76}$$

The coefficient of dx'^A in $d'\alpha'$ will be the sum of three terms, one of which is

$$(a'_{Cr},_q - a'_{Cq},_r)dx'^q \wedge dx'^C \wedge dx'^r, \tag{6.77}$$

and the other two are cyclic permutations. We partial-differentiate (6.77) with respect to dx'^p and multiply by $dx'^p \wedge dx'^B$ on the right. We proceed similarly with $i = q$ and $i = r$, and add all these contributions. We thus get

$$(a'_{Cr},_{qp} - a'_{Cq},_{rp} + a'_{Cp},_{rq} - a'_{Cr},_{pq} + a'_{Cq},_{pr} - a'_{Cp},_{qr})z'. \tag{6.78}$$

By virtue of equality of second partial derivatives, terms first, second and third inside the parenthesis cancel with terms fourth, fifth and sixth. To complete the proof, we follow the same process with another dx'^C and the same triple (p, q, r) until we exhaust all the options. We then proceed to choose another triple and repeat the same process until we are done with all the terms, which completes the proof of identical vanishing of the second term arising from one of the two integrations by parts of the previous subsection.

6.6 Hodge's Theorems

The "beyond" in the title of this chapter responds to the fact that we shall be doing much more than reproducing Hodge's theorem. As is the case with Helmholtz theorem, we are able to specify in terms of integrals what the different terms are.

We shall later embed Riemannian spaces R_n in Euclidean spaces E_N, thus becoming $n-$ surfaces. As an intermediate step, we shall apply the traditional Helmholtz approach to regions of Euclidean spaces, i.e. R_n's ab initio embedded in E_n. The harmonic form —which is of the essence in Hodge's theorem— emerges from the Helmholtz process in the new venues.

6.6.1 Transition from Helmholtz to Hodge

Though visualization is not essential to follow the argument, it helps for staying focused. For that reason, we shall argue in 3-D Euclidean space. It does not interfere with the nature of the argument.

On a region R of E_3, including the boundary, define a differential $1-$form or $2-$form α. Let A denote any continuously differentiable prolongation of α that vanish sufficiently fast at infinity. On R, we have $dA = d\alpha$ and $\delta A = \delta\alpha$. We can apply Helmholtz theorem to the differential forms A. In order to minimize clutter, we write it in the form

$$\begin{aligned} -4\pi A &= d...\int_{R'} \frac{\delta' A'...}{r_{12}} + \delta \int_{R'} ...\frac{d' A'...}{r_{12}} + \\ &\quad + d...\int_{E_3'-R'} \frac{\delta' A'...}{r_{12}} + \delta...\int_{E_R'-R'} \frac{d' A'...}{r_{12}}, \end{aligned} \tag{6.79}$$

where $r_{12} = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}$. We shall keep track of the fact, at this point obvious, that in the first two integrals on the right, r' is in R'. It is outside R' in the other two integrals, which will depend on the prolongation. By representing those terms simply as $\mathcal{F}$, we have

$$-4\pi A = d...\int_{R'} \frac{\delta' \alpha'...}{r} + \delta...\int_{R'} \frac{d' \alpha'...}{r} + \mathcal{F}. \tag{6.80}$$

Since these equations yield A everywhere in E_3 (i.e. r not limited to R), they yield in particular what A and $\mathcal{F}$ are in R . We can thus write

$$-4\pi \alpha = d...\int_{R'} \frac{\delta' \alpha'...}{r} + \delta...\int_{R'} \frac{d' \alpha'...}{r} + \mathcal{F}, \tag{6.81}$$

$\mathcal{F}$ not having changed except that $\mathcal{F}$ in (6.81) refers only to what it is in R but it remains a sum of integrals in $E_3' - R$. The prolongations will be determined as different solutions of a differential system to be obtained as follows.

By following the same process as in Helmholtz theorem, we obtain, in particular,

$$-4\pi d\alpha = d\delta...\int_{R'} \frac{d' \alpha'...}{r} + d\mathcal{F}, \tag{6.82}$$

and similarly for $-4\pi\delta\alpha$ (just exchange d and δ).

Now, the first term on the right hand side of (6.82) will not become simply $-4\pi d\alpha$ as was the case in the previous section. It will yield two terms. One of them is $-4\pi d\alpha$, and the other one is made to cancel with $d\mathcal{F}$, thus determining a differential equation to be satisfied by $\mathcal{F}$. To this we have to add another differential equation arising from application of δ to (6.81). Together they determine the differential system to be determined by $\mathcal{F}$. Thus $-4\pi\alpha$ will be given by the three term decomposition (6.81). Notice that, in the process, we avoid integrating over $E_3' - R'$ and instead solving a differential system in R, since the left hand side and the first term on the right hand side of (6.82) pertain to α.

From now one, we shall make part of the theorems that the prolongations are solutions of a certain differential systems, later to be made explicit.

6.6.2 Hodge theorem in regions of $\mathbf{E}_n$

Let α be a differential $k-$form satisfying the equations $d\alpha = \mu$ and $\delta\alpha = \nu$, and given at the boundary of a region of E_n. We proceed to integrate this system. (6.81) now reads

$$-(n-2)S_{n-1}\alpha = d\left[\int_{R'} \frac{\delta'\alpha'...}{r_{12}}\right] + \delta\left[\int_{R'} \frac{d'\alpha'...}{r_{12}}\right] + \mathcal{F}, \tag{6.83}$$

where R is a region of Euclidean space that contains the origin and where r_{12} is the magnitude of the Euclidean distance between hypothetical points of components $(x, y, ...u, v)$.and $(x', y', ...u', v')$, all the coordinates chosen as Cartesian to simplify visualization. We said hypothetical because the interpretation as distance only makes sense when we superimpose E_n and E'_n.

When we apply either d or δ to (6.83), we shall use, as before, $d\delta + \delta d = \partial\partial$, with one of the terms on the left moved to the right ($d\delta = ...$, $\delta d = ...$ respectively). By developing the $\partial\partial$ term, it becomes the same as term on the right (i.e $d\alpha$ or $\delta\alpha$). It will cancel with the term on the left. The terms that vanished identically also vanish now, precisely because this is an identical vanishing. We are thus left with the total differential terms. If apply Stokes theorem, as before. these terms no longer disappear at the boundary. Hence, we are left with the two equations

$$\left[dx^A \left({}^{\cdot}_{\wedge} \right) dx^i\right] \left({}^{\wedge}_{\cdot} \right) dx^l \int_{R'} \left(\frac{\partial \frac{1}{r_{12}^{n-2}}}{\partial x'^l} \right) dx'^i \cdot \left[\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}} \right] + \begin{pmatrix} d \\ \delta \end{pmatrix} \mathcal{F} = 0 \tag{6.84}$$

(Refer to (6.81)). Hence, the solution to Helmholtz problem is given by the pair of equations (6.83)-(6.84).

We shall now show that $\mathcal{F}$ is harmonic, i.e. $(d\delta + \delta d)\mathcal{F} = 0$..We shall apply δ and d to the first and second lines of (6.94). Start by rewriting the first terms in (6.93) in the form, (6.80), they took before applying Stoke's theorem. Upon applying the δ operator to the first line, we have, for $\delta d\mathcal{F}$,

$$dx^h \cdot \left[(dx^A \cdot dx^i) \wedge dx^l\right] \int_{R'} \frac{\partial^2}{\partial x'^h \partial x'^i} \left[\left(\frac{\partial}{\partial x'^l} \frac{1}{r_{12}^{n-2}} \right) \begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}} \right]. \tag{6.85}$$

Since this term happens to vanish, the computation will take place up to the factor -1, provided it is common to all terms in a development into explicit terms. We do so because (6.85) will be shown to vanish identically.

For $dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^l]$ to be different from zero, h and i must be different and contained in A. Since dx^l is not in dx^A, the product $dx^h \cdot dx^l$ is zero. Hence

$$dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^l] = [dx^h \cdot (dx^A \cdot dx^i)] \wedge dx^l]. \tag{6.86}$$

We can always write dx^A as

$$dx^h \wedge dx^j \wedge dx^C \wedge dx^i. \tag{6.87}$$

This is antisymmetric in the pair (i, h), which combines with the symmetry inside the integral to annul this term. Notice that we did not have to assign specific values for (i, h), but we had to "go inside" dx^A. We mention this for contrast with the contents for the next paragraph. We have proved so far that $\delta d\mathcal{F} = 0$.

We rewrite the left hand side of (6.84) as in (6.70) and proceed to apply d to it. We shall now have

$$dx^h \wedge [(dx^A \wedge dx^i) \cdot dx^l] \int_{R'} \frac{\partial^2}{\partial x'^h \partial x'^i} \left[\left(\frac{\partial}{\partial x'^l} \frac{1}{r_{12}^{n-2}} \right) d'\alpha' \wedge dx'^{\bar{A}} \right]. \tag{6.88}$$

It is clear that, when l takes a value different from the value taken by i, we again have cancellation due to the same combination of antisymmetry-symmetry as before. But the terms $dx^i \cdot dx^l$ would seem to interfere with the argument, but it does not. We simply have to be more specific than before with the groups of terms that we put together. We put together only terms where we have $dx^r \wedge dx^s$ arising from $(h = r,\ i = s)$ and $(h = s,\ i = r)$. When the running index l takes the values r or s, the resulting factor at the front of the integral will belong to a different group. We have thus shown that (6.88) cancels out and, therefore, $d\delta\mathcal{F} = 0$. To be precise, we have not only proved that $\mathcal{F}$ is harmonic, but that it is "hyper-harmonic", meaning precisely that: $\delta d\mathcal{F} = 0$ and $d\delta\mathcal{F} = 0$.

6.6.3 Hodge's theorem for hypersurfaces of $\mathbf{E}_N$

A manifold embedded in a Euclidean space of the same dimension will be called a region thereof. A hypersurface is a manifold of dimension n embedded in a Euclidean space E_N where $N > n$. The treatment here is the same as in subsection 6.1, the hypersurface playing the role of the region. The only issue that we need to deal with is a practical one having to do with the experience of readers. Helmholtz magnificent theorem belongs to an epoch where vector (and tensor) fields often took the place of differential forms. This can prompt false ideas as we now explain.

Let $\mathbf{v}$ be a vector field $\mathbf{v} \equiv a^{\lambda}(u,v)\hat{\mathbf{a}}_{\lambda}$ $(\lambda = 1,2)$ on a surface $x^{i}(u,v)$ $(i = 1,2,3)$ embedded in E_3, the frame field $\hat{\mathbf{a}}_{\lambda}$ being orthonormal. It can be tangent or not tangent. By default, the vector field is zero over the remainder of E_3. In its present form , Helmholtz theorem would not work for this field since the volume integrals over E_3 would be zero. This is a spurious implication because the theorem should be about algebras of differential forms, not tangent spaces.

Let μ be the differential $1-$form $a_{\lambda}(u,v)\hat{\omega}^{\lambda}$, the basis $\hat{\omega}^{\lambda}$ being dual to the constant orthonormal basis field $\mathbf{a}_{\lambda}$. This duality yields $a_{\lambda} = a^{\lambda}$. No specific curve is involved in the definition of μ, which is a function of curves, function determined by its coefficients $a_{\lambda}(u,v)$ The specification of a vector field on a surface, $\mathbf{v}$, on the other hand needs to make reference to a surface for its definition. And yet the components of $d\mu$ and $\delta\mu$ (which respectively are a $2-$form and a $0-$form) enter non-null volume integrals, which pertain to $3-$forms . The fact that most components (in the algebra) of an $k-$form are zero is totally irrelevant. The Helmholtz theorem for, say, a differential $1-$form μ can be formulated in any sufficiently high dimensional Euclidean space regardless of whether the "associated" vector field $\mathbf{v}$ is zero outside some surface.

Similarly, Helmholtz theorem for a differential $n-$form in E_N involves the integration of differential $N-$forms, built upon the interior differential $(n-1)-$form and the exterior differential $(n+1)-$form. In considering simple examples (say a plane in $3-$space), one can be misled or confused if one does not take into account the role of $1/r$, or else we might be obtaining an indefinite integral. Assume finite $\int \lambda(x,y)dx \wedge dy$ when integrating over the xy plane. The integral $\int \lambda(x,y)dx \wedge dy \wedge dz$ would be divergent, but need not be so if there is some factor that goes to zero sufficiently fast at infinity of z and $-z$.

6.6.4 Helmholtz-Hodge's and Hodge's theorem for Riemannian spaces

We shall consider a Helmholtz-Hodge extension of Hodge's theorem (i.e. a theorem of integration) and the standard Hodge theorem, which is a consequence of the former.

Consider now a differentiable manifold R_n endowed with a Euclidean metric. By the Schläfli-Janet-Cartan theorem [1],[2],[3], it can be embedded in a Euclidean space of dimension $N = n(n+1)/2$. Hence, a Helmholtz-Hodge theorem follows for orientable Riemannian manifolds that satisfy the conditions for application of Stokes theorem by viewing them as hypersurfaces in Euclidean spaces. At this point in our argument, the positive definiteness of the metric is required, or else we would have to find a replacement for the Laplacians considered in previous sections. The result is local, meaning non global, remark made in case the term local might send some physicists in a different direction. For clarity, the evaluation of the Laplacian now satisfies

$$1 = \frac{1}{(N-2)S_{N-1}} \int_{E_N} \partial\partial \frac{1}{r^{N-2}} z, \tag{6.89}$$

where r is the radial coordinate in $N-$dimensional space. Needless to say that it also applies to regions and hypersurfaces of E_N that contain the origin. As a consequence of the results in the previous subsections, we have the following.

Helmholtz-Hodge's theorem:

Hodge's theorem is constituted by Eqs. (6.90)-(6.91): For differential $k-$forms in Riemannian spaces R_n

$$-(N-2)S_{N-1}\alpha = d\left[\omega^A \int_{R'_n} \frac{(\delta'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{N-2}}\right] + \delta\left[\omega^A \int_{R'_n} \frac{(d'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{N-2}}\right] + \mathcal{F}, \tag{6.90}$$

$$\begin{pmatrix} d \\ \delta \end{pmatrix} \mathcal{F} = -\left[dx^A \left(\begin{smallmatrix} \cdot \\ \wedge \end{smallmatrix}\right) dx^i\right] \left(\begin{smallmatrix} \wedge \\ \cdot \end{smallmatrix}\right) dx^l \int_{R'_n} \left(\frac{\partial \frac{1}{r_{12}^{n-2}}}{\partial x'^l}\right) dx'^i \cdot \left[\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}\right], \tag{6.91}$$

with r_{12} being defined in any Euclidean space of dimension $N \geq n(n+1)/2$ where we consider R_n to be embedded.

As previously discussed, r_{12} represents a chord. We insist once more that $\omega'^{\bar{A}}$ is determined by the specific term in $\delta'\alpha'$ and $d'\alpha'$ that it multiplies. $\mathcal{F}$ is undetermined by solutions of the system $\delta\alpha = 0$, $d\alpha = 0$. So is, therefore α.

Hodge's theorem, as opposed to Helmholtz-Hodge theorem, is about decomposition. Hence, once again, uniqueness refers to something different from the uniqueness in the theorem of subsection (3.2), which refers to a differential system.

One might be momentarily tempted to now apply (6.90) to (6.91). We would get an identity, $\mathcal{F} = \mathcal{F}$, by virtue of the orthogonality of the subspace of the harmonic differential forms to the subspaces of closed and co-closed differential forms.

Hodge's theorem:

Any differential $k-$form, whether of homogeneous grade or not, can be uniquely decomposed into closed, co-closed and hyper-harmonic terms. For differential $k-$forms, the theorem is an immediate consequence of (6.100). For differential forms which are not of homogeneous grade, the theorem also applies because one only needs to add the decompositions of the theorem for the different homogeneous $k-$forms that constitute the inhomogeneous differential form.

This is obviously contained in Helmholtz-Hodge's theorem.

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